Kinks of the sine-Hilbert equation and their dynamical motions

1987 J. Phys. A: Math. Gen. 20 3587

(http://iopscience.iop.org/0305-4470/20/12/014)

J. Phys. A: Math. Gen. **20** (1987) 3587-3606. Printed in the UK

Kinks of the sine-Hilbert equation and their dynamical motions

Yoshimasa Matsuno

Department of Physics, Faculty of Liberal Arts, Yamaguchi University, Yamaguchi 753, Japan

Received 12 September 1986

Abstract. An exact method for constructing exact solutions of the sine-Hilbert (SH) equation is developed. It is shown that the SH equation can be transformed into a system of non-linear ordinary differential equations through a dependent variable transformation. Furthermore, this system of equations is transformed into a system of decoupled linear ordinary differential equations. Exact solutions of the SH equation are then constructed by means of a simple integration. The dynamical properties of the solutions are also studied in detail. The solution is interpreted as a superposition of N pulses which may be termed kinks, where N is a positive integer. Asymptotically for large time, it is composed of a pulse with a constant velocity and $N-1$ 'static pulses'. This is a novel characteristic of the solution, unlike the well known N-soliton solution which behaves like N moving pulses for large time.

1. Introduction

In recent studies of soliton theory, much attention has been paid to non-linear integro-differential equations. Among these equations, the Benjamin-Ono (BO) equation (Benjamin 1967, Ono 1975) and the intermediate long wave (ILW) equation (Joseph 1977, Joseph and Egri 1978, Kubota *et al* 1978) are typical examples. The various exact methods have been developed in order to solve these equations. The inverse scattering method (Gardner *et al* 1967, Ablowitz and Segur 1981, Calogero and Degasperis 1982, Dodd *et al* 1983) and the bilinear transformation method (Hirota 1971, Bullough and Caudrey 1980, Matsuno 1984) are powerful tools for analysing certain classes of non-linear evolution equations. Both methods have been applied successfully to the BO (Matsuno 1979a, 1980, Satsuma and Ishimori 1979, Fokas and Ablowitz 1983) and the ILW (Matsuno 1979b, Kodama *et al* 1982) equations and the N-soliton and the N-periodic wave solutions for these novel integro-differential equations are now available.

Recently, a class of non-linear integro-differential equations has been found in association with a matrix spectral problem (Degasperis and Santini 1983). The so-called sine-Hilbert (SH) equation belongs to this class of equations. In the original version of the SH equation due to Degasperis and Santini (1983), it is expressed as the following system of non-linear integro-differential equations

$$u_t = vHv \tag{1.1a}$$

$$v_t = uHv. \tag{1.1b}$$

0305-4470/87/123587+20$02.50

Here, $u=u(x, t)$ and $v=v(x, t)$ are scalar functions of two variables, x and t, the integral operator H defined by

$$Hv(x, t)=\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{v(y, t)}{y-x} \mathrm{d}y \tag{1.2}$$

is the Hilbert transform (the symbol P in (1.2) stands for the Cauchy principal value) and the subscript t appended to u and v denotes partial differentiation with respect to t.

Subsequently, Degasperis *et al* (1985) and Santini *et al* (1985) have demonstrated that the system of equations (1.1) can be associated with a Riemann-Hilbert spectral problem. These authors introduced a new dependent variable, $\theta=\theta(x, t)$, through the relations $v=\mathrm{i}\sin\theta$ and $u=(1+v^2)^{1/2}=\cos\theta$ and took account of a property of the H operator, $H^2=-1$ to recast (1.1) into the following single equation which they named the SH equation:

$$H\theta_t=-\sin\theta. \tag{1.3}$$

The SH equation is derived formally from the sine-Gordon equation, $\theta_{xt}=-\sin\theta$ if we replace the x derivative by the Hilbert transform. This formal derivation is entirely analogous to that of the BO equation from the Korteweg-de Vries equation.

The construction of pure soliton solutions has been performed by means of a Riemann-Hilbert scattering method (Santini *et al* 1985). However, these authors did not present the explicit expression of an N-soliton solution for $N\geq 2$.

The purpose of the present paper is to develop a systematic method for constructing exact non-periodic solutions of (1.3) which are real and finite over all x and t and to analyse the properties of the solutions in detail. The method of construction developed in this paper stems quite naturally from the bilinear transformation method introduced by Hirota (1971). This paper also serves to amplify the results of a previous note (Matsuno 1986). Finally, it should be remarked that the periodic solutions of (1.3) which reduce, in an appropriate limit, to the non-periodic solutions presented here were constructed quite recently (Matsuno 1987).

In § 2, we show that the SH equation can be transformed into a system of non-linear ordinary differential equations through a dependent variable transformation. At the same time, it is shown that this system of equations is equivalent to a bilinear equation. The system of equations is furthermore converted into a system of decoupled *linear* ordinary differential equations. Exact solutions of (1.3) are then constructed by a simple integration. In § 3, the properties of the solutions are studied in detail. The solution presented is interpreted as a superposition of N pulses, where N is positive integer. However, as a result of interactions between pulses, the solution is shown to be composed of a pulse moving in the positive x direction with a constant velocity and $N-1$ 'static pulses' with very narrow widths in the limit of large time. This is a remarkable property of solutions which is quite unlike the asymptotic behaviour of the usual N-soliton solution. Section 4 is devoted to concluding remarks.

2. Method for exact solution

In this section, we shall develop a systematic method for constructing exact solutions of (1.3) which are real and finite over all x and t and satisfy the boundary conditions $\theta_x\to 0$ as $x\to\pm\infty$.

First of all, let us introduce the following dependent variable transformation

$$\theta = \mathrm{i} \ln(f^*/f) \tag{2.1}$$

$$f = f(x, t) = \prod_{j=1}^{N} (x - x_j(t)) \tag{2.2a}$$

$$\operatorname{Im} x_j(t) > 0 \qquad j = 1, 2, \ldots, N \qquad x_n \neq x_m \text{ for } n \neq m \tag{2.2b}$$

where x_j ($j = 1, 2, \ldots, N$) are complex functions of t and * denotes complex conjugate†. Then, it readily follows from (1.2), (2.1) and (2.2) that

$$\begin{aligned} H\theta_t &= \mathrm{i} \sum_{j=1}^{N} \left[\dot{x}_j H\left(\frac{1}{x - x_j}\right) - \dot{x}_j^* H\left(\frac{1}{x - x_j^*}\right) \right] \\ &= \sum_{j=1}^{N} \left(\frac{\dot{x}_j}{x - x_j} + \frac{\dot{x}_j^*}{x - x_j^*} \right) \\ &= -(\ln(f^* f))_t \end{aligned} \tag{2.3}$$

where we have used the formula

$$H\left(\frac{1}{x - x_j}\right) = \frac{-\mathrm{i}}{x - x_j} \qquad \operatorname{Im} x_j > 0. \tag{2.4}$$

In (2.3), the dot appended to x_j and x_j^* means differentiation with respect to t. Substituting (2.1), (2.2) and (2.3) into (1.3), equation (1.3) is transformed into the form

$$\sum_{j=1}^{N} \left(\frac{\dot{x}_j}{x - x_j} + \frac{\dot{x}_j^*}{x - x_j^*} \right) = \frac{1}{2\mathrm{i}} \left(\frac{\Pi_{j=1}^{N} (x - x_j^*)}{\Pi_{j=1}^{N} (x - x_j)} - \frac{\Pi_{j=1}^{N} (x - x_j)}{\Pi_{j=1}^{N} (x - x_j^*)} \right). \tag{2.5}$$

It should be noted that (2.5) is equivalent to the following bilinear equation:

$$(f^* f)_t = \frac{1}{2\mathrm{i}} (f^2 - f^{*2}) = \operatorname{Im} f^2. \tag{2.6}$$

Multiplying both sides of (2.5) by $x - x_n$ ($n = 1, 2, \ldots, N$) and then putting $x = x_n$, we obtain a system of non-linear ordinary differential equations for x_n as follows:

$$\dot{x}_n = \frac{1}{2\mathrm{i}} \frac{\Pi_{j=1}^{N} (x_n - x_j^*)}{\Pi_{j=1 (j \neq n)}^{N} (x_n - x_j)} \qquad n = 1, 2, \ldots, N. \tag{2.7}$$

Therefore, in the present situation the problem considered here has been reduced to solve dynamical motions of N variables, $x_1, x_2, \ldots, x_N$.

We shall now show that the system of equations (2.7) can be linearised by an appropriate procedure and motions of $x_1, x_2, \ldots, x_N$ are determined by solving an algebraic equation of order N. It will be demonstrated, however, that the solutions of (1.3) can be constructed without solving the algebraic equation explicitly.

† It should be remarked that the dependent variable transformation for θ_x is the same as that used for the BO equation (Matsuno 1979a, 1980).

In the process of linearisation, it is a key point to introduce the following polynomials s_n and p_n $(n=1,2,\ldots,N)$:

$$s_1=\sum_{j=1}^{N} x_j \tag{2.8a}$$

$$s_2=\sum_{j<k}^{N} x_j x_k, \tag{2.8b}$$

$$\vdots$$

$$s_N=\prod_{j=1}^{N} x_j \tag{2.8c}$$

$$p_n=\sum_{j=1}^{N} x_j^n \qquad n=1,2,\ldots,N \tag{2.9}$$

where s_n $(n=1,2,\ldots,N)$ are elementary symmetric functions of $x_1, x_2, \ldots, x_N$. As is well known, the p_n and s_n are related by Euler's formula

$$p_n=\sum_{j=1}^{n-1}(-1)^{n-1-j}s_{n-j}p_j+(-1)^{n-1}ns_n \qquad n=2,3,\ldots,N \tag{2.10a}$$

$$p_1=s_1. \tag{2.10b}$$

A few of p_n are given by

$$p_2=-2s_2+s_1^2 \tag{2.11a}$$

$$p_3=3s_3-3s_1s_2+s_1^3 \tag{2.11b}$$

$$p_4=-4s_4+4s_1s_3-4s_1^2s_2+2s_2^2+s_1^4. \tag{2.11c}$$

As the first step for constructing solutions of (1.3), we shall derive the time evolution of p_n. Before entering into detail, it is useful to note the following formulae:

$$\sum_{n=1}^{N}\frac{x_n^k}{\prod_{j=1(j\neq n)}^{N}(x_n-x_j)}=\delta_{k,N-1} \qquad k=0,1,\ldots,N-1 \tag{2.12}$$

$$\frac{x^{N+m}}{\prod_{j=1}^{N}(x-x_j)}=\sum_{j=0}^{m}c_{m-j}x^j+\frac{\sum_{j=0}^{N-1}d_j^{(m)}x^j}{\prod_{j=1}^{N}(x-x_j)} \tag{2.13}$$

where $\delta_{k,N-1}$ is Kronecker's delta function and c_{m-j} and $d_j^{(m)}$ in (2.13) are determined successively by the following recursion relations:

$$\sum_{j=0}^{n}(-1)^{n-j}c_j s_{n-j}=0 \qquad n=1,2,\ldots,m \tag{2.14a}$$

$$\sum_{j=0}^{m}(-1)^{n-j}c_j s_{n-j}+d_{N+m-n}^{(m)}=0 \qquad n=m+1,m+2,\ldots,m+N \tag{2.14b}$$

$$c_0=s_0=1. \tag{2.14c}$$

Explicitly, the first few of c_j are written as

$$c_1=s_1 \tag{2.15a}$$

$$c_2=-s_2+s_1^2 \tag{2.15b}$$

$$c_3=s_3-2s_1s_2+s_1^3 \tag{2.15c}$$

$$c_4=-s_4+2s_1s_3-3s_1^2s_2+s_2^2+s_1^4. \tag{2.15d}$$

The relation (2.12) is a consequence of Lagrange's interpolation formula. It can be proved easily by considering the contour integral

$$\frac{1}{2\pi i}\oint \frac{z^k}{\Pi_{j=1}^N (z-x_j)}\,dz$$

where the integration is performed along a closed contour containing all the x_j.

It immediately follows from (2.12), (2.13) and (2.14*b*) that

$$\sum_{n=1}^N \frac{x_n^{N+m}}{\Pi_{j=1(j\neq n)}^N (x_n-x_j)} = d_{N-1}^{(m)} = \sum_{j=0}^m (-1)^{m-j} c_j s_{m+1-j} \qquad m=0,1,\ldots,N-1. \tag{2.16}$$

Under these preparations, it is straightforward to derive the equation which governs the time evolution of p_{n+1}. We first use (2.7) and (2.12) to obtain

$$\begin{aligned}\frac{1}{m+1}\dot{p}_{m+1} &= \sum_{n=1}^N x_n^m \dot{x}_n \\ &= \frac{1}{2i}\sum_{k=0}^{m+1} (-1)^k s_k^* \sum_{n=1}^N \frac{x_n^{N+m-k}}{\Pi_{j=1(j\neq n)}^N (x_n-x_j)}.\end{aligned} \tag{2.17}$$

Owing to (2.12) and (2.16), (2.17) becomes

$$\begin{aligned}\frac{1}{m+1}\dot{p}_{m+1} &= \frac{1}{2i}\left((-1)^{m+1}s_{m+1}^* + \sum_{k=0}^m s_{m-k}^* \sum_{j=0}^k (-1)^{m-j} c_j s_{k+1-j}\right) \\ &= \frac{1}{2i}\left((-1)^{m+1}s_{m+1}^* + \sum_{j=0}^m (-1)^{m-j} c_j s_{m+1-j}\right. \\ &\quad \left. + \sum_{k=0}^{m-1} s_{m-k}^* \sum_{j=0}^k (-1)^{m-j} c_j s_{k+1-j}\right).\end{aligned} \tag{2.18}$$

On the other hand, we notice the following relation which stems from (2.14*a*) with $n=k+1$ $(k=0,1,\ldots,m-1)$

$$\sum_{j=0}^k (-1)^j c_j s_{k+1-j} = (-1)^k c_{k+1}. \tag{2.19}$$

Substituting (2.19) into the third term on the right-hand side of (2.18), one arrives at the final result

$$\dot{p}_{m+1} = \frac{(-1)^m(m+1)}{2i}\sum_{k=0}^m (-1)^k (s_{m+1-k} - s_{m+1-k}^*) c_k \qquad m=0,1,\ldots,N-1. \tag{2.20}$$

The second step, which is most important in this paper, is to derive the time evolution of s_j $(j=1,2,\ldots,N)$. It is expressed simply as

$$\dot{s}_j = \frac{1}{2i}(s_j - s_j^*) = \mathrm{Im}\, s_j \qquad j=1,2,\ldots,N. \tag{2.21}$$

In contrast to (2.7), (2.21) is a system of decoupled *linear* ordinary differential equations and the integration of which can be readily performed. We shall now prove (2.21) by a mathematical induction. Equation (2.21) with $j=1$ holds obviously because of (2.20) with $m=0$ and $p_1=s_1$ (see (2.10*b*)). Assume (2.21) for $j=1,2,\ldots,m$, i.e.

$$\dot{s}_j = \frac{1}{2i}(s_j - s_j^*) \qquad j=1,2,\ldots,m. \tag{2.22}$$

Then, it follows from (2.10*a*), (2.20) and (2.22) that

$$\dot{s}_{m+1}=\frac{(-1)^m}{m+1}\dot{p}_{m+1}+\frac{1}{m+1}\sum_{j=0}^{m-1}(-1)^j(\dot{s}_{m-j}p_{j+1}+s_{m-j}\dot{p}_{j+1})$$

$$=\frac{1}{2\mathrm{i}}(s_{m+1}-s^*_{m+1})-\frac{1}{2\mathrm{i}}\sum_{k=0}^{m-1}(-1)^k(s_{m-k}-s^*_{m-k})\left(c_{k+1}-\frac{1}{m+1}p_{k+1}\right)$$

$$+\frac{1}{m+1}\sum_{j=0}^{m-1}(-1)^j s_{m-j}\dot{p}_{j+1}. \qquad (2.23)$$

Furthermore, one obtains from (2.19) and (2.20)

$$\sum_{j=0}^{m-1}(-1)^j s_{m-j}\dot{p}_{j+1}$$

$$=\frac{1}{2\mathrm{i}}\sum_{j=0}^{m-1}(j+1)s_{m-j}\sum_{k=0}^{j}(-1)^k(s_{j+1-k}-s^*_{j+1-k})c_k$$

$$=\frac{1}{2\mathrm{i}}\sum_{j=0}^{m-1}(-1)^j(j+1)s_{m-j}c_{j+1}-\frac{1}{2\mathrm{i}}\sum_{j=0}^{m-1}(j+1)s_{m-j}\sum_{k=0}^{j}(-1)^k s^*_{j+1-k}c_k$$

$$=\frac{1}{2\mathrm{i}}\sum_{j=0}^{m-1}(-1)^j(j+1)s_{m-j}c_{j+1}$$

$$-\frac{1}{2\mathrm{i}}\sum_{j=0}^{m-1}s^*_{m-j}\left(\sum_{k=0}^{j}(-1)^k k s_{j-k+1}c_k+(-1)^j(m-j)c_{j+1}\right). \qquad (2.24)$$

Substituting (2.24) into (2.23), we find

$$\dot{s}_{m+1}=\frac{1}{2\mathrm{i}}(s_{m+1}-s^*_{m+1})+\frac{1}{2\mathrm{i}}\frac{1}{m+1}\sum_{k=0}^{m-1}(-1)^k s_{m-k}[-(m-k)c_{k+1}+p_{k+1}]+\frac{1}{2\mathrm{i}}\frac{1}{m+1}$$

$$\times\sum_{k=0}^{m-1}(-1)^k s^*_{m-k}\left((k+1)c_{k+1}-p_{k+1}-(-1)^k\sum_{j=0}^{k}(-1)^j j s_{k-j+1}c_j\right). \qquad (2.25)$$

On the other hand, we see from (2.10*a*) and (2.19)

$$(k+1)c_{k+1}-p_{k+1}=\sum_{j=0}^{k-1}(-1)^{k+j}s_{k-j}[p_{j+1}-(k+1)c_{j+1}]. \qquad (2.26)$$

Introducing (2.26) into the third term on the right-hand side of (2.25), we obtain

$$\dot{s}_{m+1}=\frac{1}{2\mathrm{i}}(s_{m+1}-s^*_{m+1})-\frac{1}{2\mathrm{i}}\frac{1}{m+1}\sum_{k=0}^{m-1}(-1)^k s_{m-k}[(m-k)c_{k+1}-p_{k+1}]$$

$$-\frac{1}{2\mathrm{i}}\frac{1}{m+1}\sum_{k=1}^{m-1}s^*_{m-k}\sum_{j=0}^{k-1}(-1)^j s_{k-j}[(k-j)c_{j+1}-p_{j+1}]. \qquad (2.27)$$

However, as shown in the appendix, the following relation holds:

$$\sum_{j=0}^{k-1}(-1)^j s_{k-j}[(k-j)c_{j+1}-p_{j+1}]=0 \qquad k=1,2,\ldots,m. \qquad (2.28)$$

Hence, the second and third terms on the right-hand side of (2.27) vanish and (2.27) becomes

$$\dot{s}_{m+1} = \frac{1}{2\mathrm{i}}(s_{m+1} - s^*_{m+1}) \tag{2.29}$$

which implies that (2.22) holds for $j = m+1$, completing the proof of (2.21).

The final step is to construct solutions of (1.3) and this can be done quite easily. Integrating (2.21) yields immediately

$$s_j = a_j t + b_j + \mathrm{i} a_j \qquad j = 1, 2, \ldots, N \tag{2.30}$$

where a_j and b_j are real constant. Using (2.2a), (2.8) and (2.30), we find the explicit expression of f as follows:

$$\begin{aligned} f &= \prod_{j=1}^{N} (x - x_j) \\ &= \sum_{j=0}^{N} (-1)^j s_j x^{N-j} \\ &= x^N + \sum_{j=1}^{N} (-1)^j (a_j t + b_j + \mathrm{i} a_j) x^{N-j} \end{aligned} \tag{2.31}$$

$$\mathrm{Re}\, f = x^N + \sum_{j=1}^{N} (-1)^j (a_j t + b_j) x^{N-j} \tag{2.32}$$

$$\mathrm{Im}\, f = \sum_{j=1}^{N} (-1)^j a_j x^{N-j}. \tag{2.33}$$

We can also confirm by direct calculation that (2.31) indeed satisfies the bilinear equation for f (2.6).

The time evolutions of $x_1, x_2, \ldots, x_N$ are given by the roots of the algebraic equation of order N, $f = 0$. However, it is unnecessary to solve the algebraic equation explicitly to construct the solution as already shown by (2.1) and (2.31). This fact should be remarkable since it is difficult in general to solve the algebraic equation of an arbitrary order without recourse to numerical calculations. If we note the relation

$$\mathrm{i} \ln(f^*/f) = 2 \tan^{-1}(\mathrm{Im}\, f / \mathrm{Re}\, f) \tag{2.34}$$

we can express the solution of (1.3) in terms of only real quantities. Thus, we obtain from (2.1), (2.32) and (2.33) the explicit exact solution of (1.3) as follows:

$$\theta = 2 \tan^{-1} \frac{\Sigma_{j=1}^{N} (-1)^j a_j x^{N-j}}{x^N + \Sigma_{j=1}^{N} (-1)^j (a_j t + b_j) x^{N-j}}. \tag{2.35}$$

The solution (2.35) is real and finite for all x and t and behaves asymptotically for large x as†

$$\theta \sim -\frac{2a_1}{x} + \mathrm{O}(x^{-2}) \qquad x \to \pm\infty. \tag{2.36}$$

The asymptotic behaviour for large t will be discussed in detail in the next section.

† The principal value of θ has been taken. This convention will be used in the next section.

3. Properties of solutions

3.1. Special case

In this section, we shall study the properties of the solution (2.35) in detail. Before discussing the properties for general N, we consider two special cases of $N=1, 2$. For $N=1$, (2.35) gives

$$\theta = -2\tan^{-1}\frac{a_1}{x-a_1t-b_1} \tag{3.1}$$

$$a_1 > 0 \tag{3.2}$$

which represents a pulse moving in the positive x direction with a velocity a_1 and a centre position being placed on $x=a_1t+b_1$. The condition (3.2) is necessary in order to satisfy $\mathrm{Im}\, x_1 > 0$ (see (2.2b)). Differentiating (3.1) with respect to x yields

$$\theta_x = \frac{2a_1}{(x-a_1t-b_1)^2+a_1^2}. \tag{3.3}$$

It is interesting to note that this functional form coincides with the one-soliton solution of the BO equation (Benjamin 1967, Ono 1975)†. Figure 1 depicts a profile of the solution (3.1) where the principal value of θ has been taken, i.e. the range of θ has been restricted by $-\pi \leqslant \theta \leqslant \pi$.

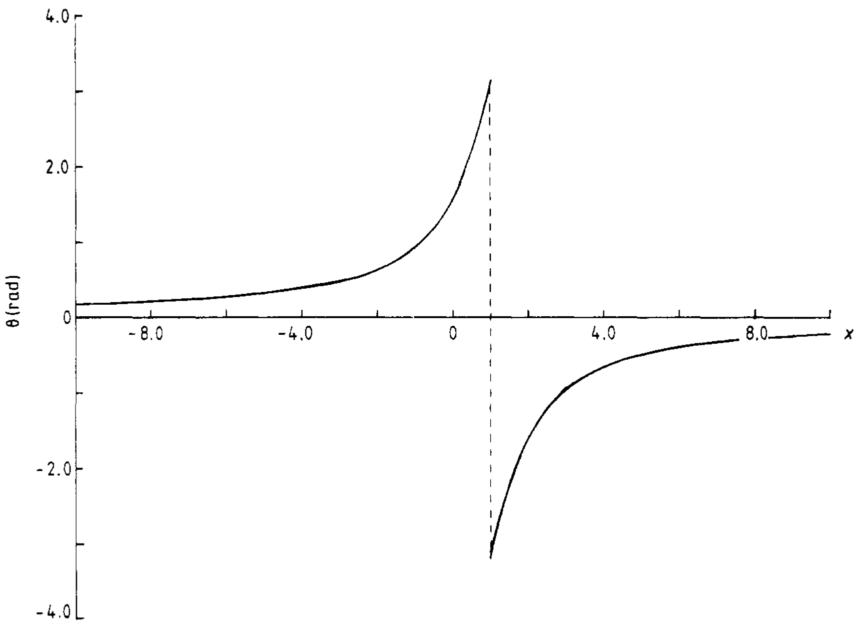

Figure 1. Profile of the solution (3.1).

† Strictly speaking, the one-soliton solution of the BO equation, $u_t+2uu_x+Hu_{xx}=0$, is expressed in the form $u=2a_1[(x-a_1^{-1}t-b_1)^2+a_1^2]^{-1}$. Hence, it is necessary to introduce a scale transformation, $t\to a_1^{-2}t$ in (3.3) in order that both functions coincide exactly.

For $N=2$, (2.35) takes the form of

$$\theta = -2\tan^{-1}\frac{a_1x-a_2}{(x-x_+)(x-x_-)} \tag{3.4}$$

where

$$x_\pm = \tfrac{1}{2}\{a_1t+b_1\pm[(a_1t)^2+(2a_1b_1-4a_2)t+b_1^2-4b_2]^{1/2}\} \tag{3.5}$$

$$a_1>0 \tag{3.6a}$$

$$a_1a_2b_1>a_1^2b_2+a_2^2. \tag{3.6b}$$

The time dependences of x_1 and x_2 are obtained by solving the algebraic equation of order two, $f=0$, where f is given by (2.31) with $N=2$. The result is expressed as

$$x_1=\frac{1}{2}\left\{A+\frac{D}{|D|}\left(\frac{C+(C^2+D^2)^{1/2}}{2}\right)^{1/2}+\mathrm{i}\left[B+\left(\frac{-C+(C^2+D^2)^{1/2}}{2}\right)^{1/2}\right]\right\} \tag{3.7a}$$

$$x_2=\frac{1}{2}\left\{A-\frac{D}{|D|}\left(\frac{C+(C^2+D^2)^{1/2}}{2}\right)^{1/2}+\mathrm{i}\left[B-\left(\frac{-C+(C^2+D^2)^{1/2}}{2}\right)^{1/2}\right]\right\} \tag{3.7b}$$

where

$$A=a_1t+b_1 \tag{3.8a}$$

$$B=a_1 \tag{3.8b}$$

$$C=(a_1t)^2+(2a_1b_1-4a_2)t-a_1^2-4b_2+b_1^2 \tag{3.8c}$$

$$D=2a_1^2t+2a_1b_1-4a_2. \tag{3.8d}$$

Asymptotic forms of x_+, x_-, x_1 and x_2 for large t are given, respectively, as follows.

For $t\to-\infty$

$$x_+\sim\frac{a_2}{a_1}-\frac{a_1a_2b_1-a_1^2b_2-a_2^2}{a_1^3}t^{-1} \tag{3.9a}$$

$$x_-\sim a_1t+b_1-\frac{a_2}{a_1} \tag{3.9b}$$

$$x_1\sim a_1t+b_1-\frac{a_2}{a_1}+\mathrm{i}a_1 \tag{3.9c}$$

$$x_2\sim\frac{a_2}{a_1}-\frac{a_1a_2b_1-a_1^2b_2-a_2^2}{a_1^3}t^{-1}+\mathrm{i}\frac{a_1a_2b_1-a_1^2b_2-a_2^2}{a_1^3}t^{-2} \tag{3.9d}$$

and for $t\to+\infty$

$$x_+\sim a_1t+b_1-\frac{a_2}{a_1} \tag{3.10a}$$

$$x_-\sim\frac{a_2}{a_1}-\frac{a_1a_2b_1-a_1^2b_2-a_2^2}{a_1^3}t^{-1} \tag{3.10b}$$

$$x_1\sim a_1t+b_1-\frac{a_2}{a_1}+\mathrm{i}a_1 \tag{3.10c}$$

$$x_2\sim\frac{a_2}{a_1}-\frac{a_1a_2b_1-a_1^2b_2-a_2^2}{a_1^3}t^{-1}+\mathrm{i}\frac{a_1a_2b_1-a_1^2b_2-a_2^2}{a_1^3}t^{-2}. \tag{3.10d}$$

It is observed from (3.9) and (3.10) that x_1 and x_2 have the same asymptotic forms when $t \to \pm\infty$ while x_+ and x_- exchange their asymptotic forms. The conditions (3.6a) and (3.6b) are required by (2.2b) with $N=2$. A novel aspect of (3.4) is that it has one immovable point, $x = a_2/a_1$. Indeed, we can show from (3.5) and (3.6) that

$$x_-(t) < a_2/a_1 < x_+(t) \tag{3.11}$$

for all t, see figure 2. Furthermore, it is easy to see from (3.5) and (3.6) that the following inequalities always hold:

$$\dot{x}_\pm > 0. \tag{3.12}$$

The profiles of $\dot{x}_\pm$ are drawn in figure 3.

The asymptotic behaviour of (3.4) for $t \to \pm\infty$ is found from (3.9a, b) and (3.10a, b) as

$$\theta \sim -2\tan^{-1}[a_1/(x - a_1 t - b_1 + a_2/a_1)] - 2\tan^{-1}[\delta t^{-2}/(x - a_2/a_1 + \delta t^{-1})] \tag{3.13a}$$

with

$$\delta = \frac{a_1 a_2 b_1 - a_1^2 b_2 - a_2^2}{a_1^3}. \tag{3.13b}$$

The first term on the right-hand side of (3.13a) represents a pulse moving in the positive x direction with a velocity a_1 and it has the same functional form as (3.1) except for

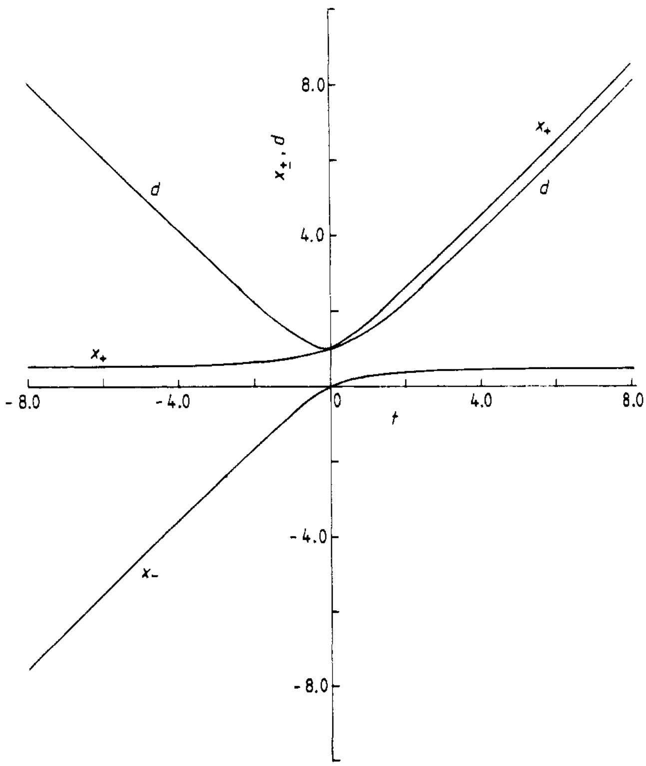

Figure 2. Plot of $x_\pm$ and d $(\equiv x_+ - x_-)$ as a function of t. The values of parameters are chosen as $a_1 = 1.0$, $a_2 = 0.5$, $b_1 = 1.0$ and $b_2 = 0$. In this case, the distance d between x_+ and x_- becomes a minimum when $t = 0$.

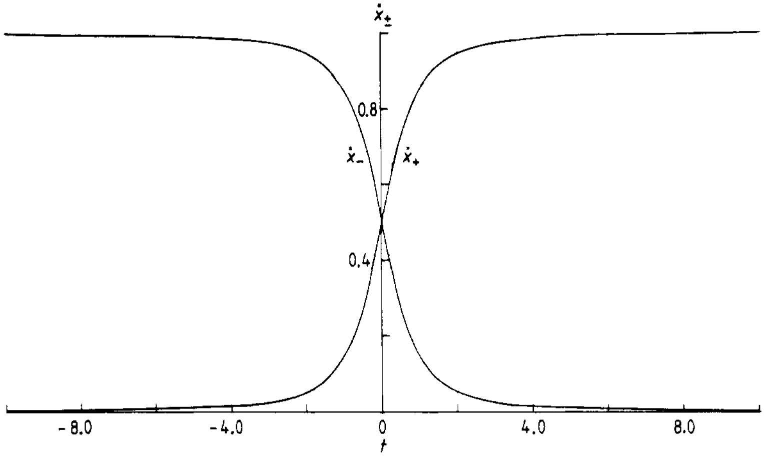

Figure 3. Plot of $\dot{x}_\pm$ as a function of t. The values of parameters are the same as those for figure 2. The $\dot{x}_+(\dot{x}_-)$ approaches to zero indefinitely when $t \to -\infty$ $(t \to +\infty)$.

a 'phase shift', $-a_2/a_1$. On the other hand, the second term represents a pulse with a very narrow width of the order of δt^{-2} and the width becomes narrower and narrower as the time goes on. It is almost static since the velocity of the pulse is given by δt^{-2} and approaches to zero indefinitely when $t \to \pm\infty$. Hence, we shall call this pulse the 'static pulse' in the following. The profile of the solution is depicted in figures 4(a)-(c) for different times. Thus, one can describe the profile of the solution (3.4) as follows.

For $t \to -\infty$, the solution is composed of a pulse moving in the positive x direction with a velocity a_1 and a static pulse with a very narrow width located near $x = a_2/a_1$ (see figure 4(a)). As time goes on, the static pulse grows wider and wider and eventually behaves like a pulse moving in the positive x direction with a velocity $\dot{x}_+$ while a moving pulse behaves like a pulse with a velocity $\dot{x}_-$ (see figure 4(b)). As can easily be seen from (3.5), the distance between the centre positions of two pulses, i.e. $d \equiv x_+ - x_-$, becomes a minimum when $t = 2(2a_2 - a_1 b_1)/a_1^2$ and it takes the value $d_{\min} = 2(a_1\delta)^{1/2}$ (see figure 2). For $t \to +\infty$, a pulse with a velocity $\dot{x}_+$ behaves like a pulse with a constant velocity a_1. On the other hand, another pulse with a velocity $\dot{x}_-$ approaches a point a_2/a_1 indefinitely and eventually becomes a static pulse with a very narrow width (see figure 4(c)).

3.2. General case

Next, we shall proceed to investigate the properties of (2.35) for an arbitrary positive integer N. For this purpose, we first examine the motion of the imaginary part of x_n to derive the necessary conditions for (2.2b). The equation of motion of Im x_n is found from (2.7) as

$$\begin{aligned} \mathrm{Im}\,\dot{x}_n &= \frac{1}{2\mathrm{i}}\left[\prod_{\substack{j=1\\(j\neq n)}}^{N}\left(1+\frac{2\mathrm{i}\,\mathrm{Im}\,x_j}{x_n-x_j}\right) - \prod_{\substack{j=1\\(j\neq n)}}^{N}\left(1-\frac{2\mathrm{i}\,\mathrm{Im}\,x_j^*}{x_n^*-x_j^*}\right)\right]\mathrm{Im}\,x_n \\ &\equiv G(t)\,\mathrm{Im}\,x_n \qquad n=1,2,\ldots,N. \end{aligned} \tag{3.14}$$

Here, $G(t)$ is a real function of t defined by the first line of (3.14). Integrating (3.14)

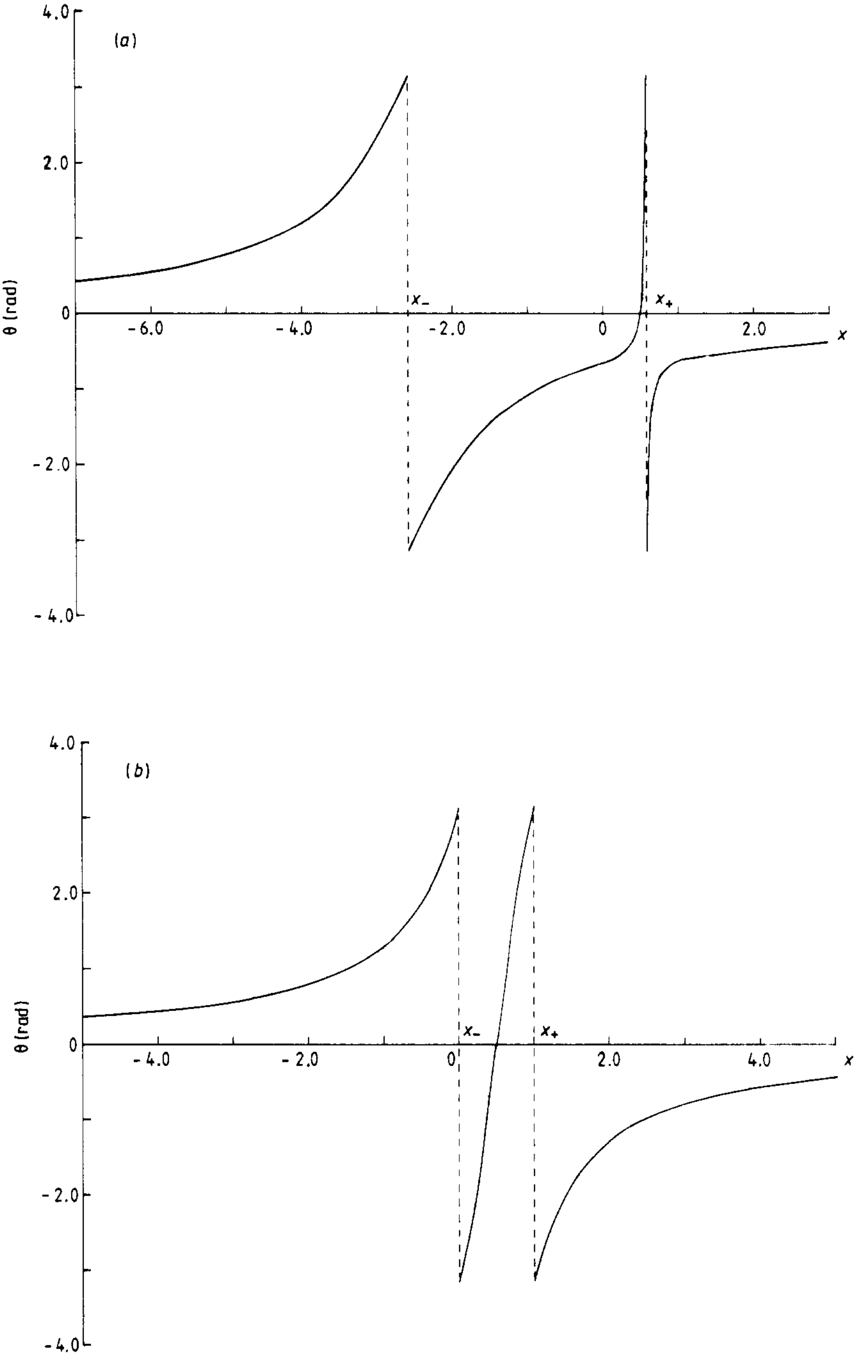

Figure 4. Profile of the solution (3.4) for three different values of t. The values of parameters are the same as those for figure 2. In the figure, the positions of $x_{\pm}$ are indicated by broken lines. (a) $t=-3.0$, (b) $t=0$, (c) $t=3.0$.

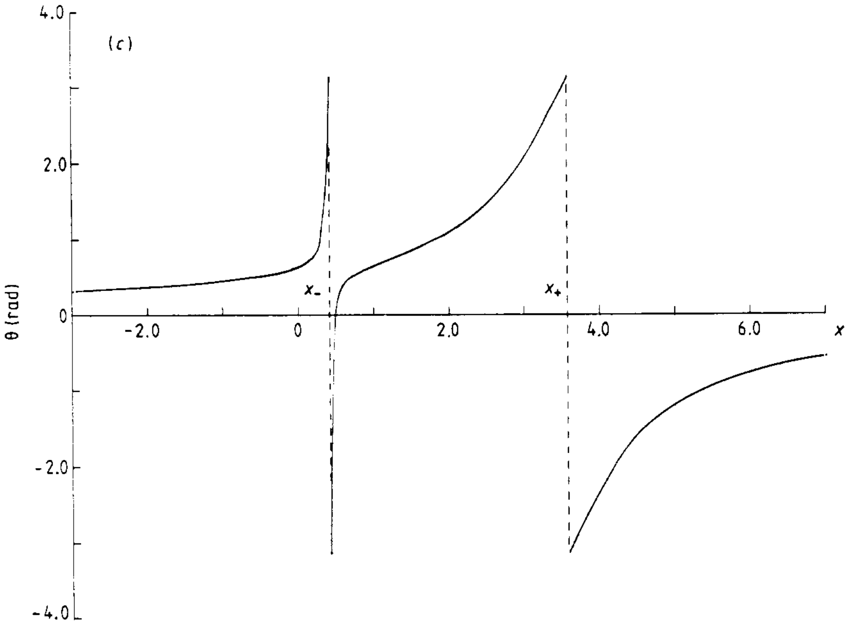

Figure 4. (continued)

now yields

$$\mathrm{Im}\, x_n(t) = \mathrm{Im}\, x_n(t_0) \exp\left(\int_{t_0}^{t} G(t')\, \mathrm{d}t'\right) \qquad n = 1, 2, \ldots, N \tag{3.15}$$

where t_0 is an arbitrary integration constant. Equations (3.15) ensure that if Im $x_n(t_0)$ ($n = 1, 2, \ldots, N$) are positive for certain t_0, then Im $x_n(t) > 0$ ($n = 1, 2, \ldots, N$) hold for all t†. In order to simplify the discussion, it is convenient to take t_0 as a very large value. Therefore, we may investigate the asymptotic solutions of the algebraic equation

$$f = 0 \tag{3.16}$$

for large t, where f is given by (2.31). The result for $N = 1, 2$ suggests that an asymptotic solution of (3.16) will be

$$x_1 \sim a_1 t + b_1 - a_2/a_1 + \mathrm{i}a_1 \qquad t \to \pm\infty. \tag{3.17}$$

To confirm this statement, let us introduce a moving coordinate ξ

$$\xi = x - a_1 t - b_1 \tag{3.18}$$

and take the limits $t \to \pm\infty$ with ξ keeping finite. We find from (2.31) in these limits

$$f \sim a_1^{N-1}(\xi - \mathrm{i}a_1 + a_2/a_1)t^{N-1} + \mathrm{O}(t^{N-2}) \tag{3.19}$$

† In a special situation, $G(t)$ has singularities. For instance, in the case of $N = 2$, setting $a_1 b_1 = 2a_2$ and $a_1^2 + 4b_2 = b_1^2$ in (3.7a) and (3.7b), which are consistent with (3.6), yields $x_1(t) = x_2(t)$ at $t = 0$. However, a careful investigation of the behaviour of x_1 and x_2 near $t = 0$ gives an estimate, $|x_1(t) - x_2(t)| \sim \mathrm{O}(|t|^{1/2})$ and therefore $|G(t)| \sim \mathrm{O}(|t|^{-1/2})$. However, this singularity is harmless since it is integrable, namely, the integral $\int_{t_0}^{t} G(t')\, \mathrm{d}t'$ has finite value at $t = 0$. For general N, the situation would be the same as that for the case of $N = 2$ since only a two-body interaction acts among x_n ($n = 1, 2, \ldots, N$) as seen from the equation of motion for x_n. One can observe that the velocity, Im $\dot{x}_n$ becomes slower and slower when Im x_n approaches to zero indefinitely and therefore the sign of Im x_n is unchanged throughout the motion.

which leads, by (2.1) and (3.18), to the expression

$$\theta \sim \mathrm{i}\ln[(x-a_1t-b_1+a_2/a_1+\mathrm{i}a_1)/(x-a_1t-b_1+a_2/a_1-\mathrm{i}a_1)]. \tag{3.20}$$

Hence, (3.17) results from (3.16) and (3.19). At the same time, the asymptotic form (3.20) implies that the imaginary parts of other $(N-1)$ solutions of (3.15) would approach to zero indefinitely. This statement holds for $N=2$ as seen from (3.9c, d) and (3.10c, d). Suggested by these facts, we seek the asymptotic solutions of (3.16) for large t in the form

$$x=A_1+A_2t^{-1}+\ldots+\mathrm{i}(B_1t^{-1}+B_2t^{-2}+\ldots) \tag{3.21}$$

where A_1, A_2, B_1 and B_2 are unknown real coefficients. Substituting (3.21) into (3.16), we find that these coefficients are determined by the following equations:

$$V(A_1)\equiv\sum_{j=1}^{N-1}(-1)^j a_j A_1^{N-j}+(-1)^N a_N=0 \tag{3.22}$$

$$B_1=0 \tag{3.23}$$

$$B_2=-A_2=\frac{A_1^N+\Sigma_{j=1}^N(-1)^j b_j A_1^{N-j}}{\Sigma_{j=1}^{N-1}(-1)^j(N-j)a_jA_1^{N-j-1}}. \tag{3.24}$$

Therefore, the necessary conditions for $\mathrm{Im}\, x_j(t)>0$, $(j=1, 2, \ldots, N)$, $(t\to\pm\infty)$ are expressed as follows:

$$a_1>0 \tag{3.25a}$$

A_1 are real and distinct simple zeros of (3.22) (3.25b)

$$B_2>0. \tag{3.25c}$$

Let α_j $(j=1,2,\ldots,N-1)$ be $N-1$ real and distinct simple zeros of (3.22) and define P_n $(n=1,2,\ldots,2N-4)$

$$P_n=\sum_{j=1}^{N-1}\alpha_j^n \qquad n=1,2,\ldots,2N-4 \tag{3.26}$$

then (3.25b) is equivalent to the following conditions (Takagi 1965):

$$D_k\equiv\begin{vmatrix} P_0 & P_1 & \ldots & P_{k-1}\\ P_1 & P_2 & \ldots & P_k\\ \vdots & \vdots & & \vdots\\ P_{k-1} & P_k & \ldots & P_{2k-2}\end{vmatrix}>0 \qquad k=1,2,\ldots,N-1. \tag{3.27}$$

The P_n are determined uniquely in terms of a_1, $a_2,\ldots,a_N$ by Euler's formula (see (2.8)-(2.10)). It should be noted that the denominator of (3.24) never vanishes due to the condition (3.25b), i.e.

$$\begin{aligned}\sum_{j=1}^{N-1}(-1)^j(N-j)a_j\alpha_n^{N-j-1}&=\left.\frac{\mathrm{d}V(x)}{\mathrm{d}x}\right|_{x=\alpha_n}\\ &=-a_1\prod_{\substack{j=1\\(j\neq n)}}^{N-1}(\alpha_n-\alpha_j)\\ &\neq 0 \qquad n=1,2,\ldots,N-1.\end{aligned} \tag{3.28}$$

We shall write down the conditions (3.25b) and (3.25c) explicitly in the case of $N=3$ for reference (see (3.6b) for $N=2$):

$$\left(\frac{a_2}{a_1}\right)^2 > 4\frac{a_3}{a_1} \tag{3.29a}$$

$$a_1 a_3 + a_1 a_2 b_1 > a_1^2 b_2 + a_2^2 \tag{3.29b}$$

$$-\left[\left(\frac{a_2}{a_1}\right)^2 - 4\frac{a_3}{a_1}\right]^{1/2} < -\frac{a_2}{a_1} + \frac{2(a_2 a_3 - a_1 a_3 b_1 + a_1^2 b_3)}{a_1 a_3 + a_1 a_2 b_1 - a_1^2 b_2 - a_2^2}$$
$$< \left[\left(\frac{a_2}{a_1}\right)^2 - 4\frac{a_3}{a_1}\right]^{1/2}. \tag{3.29c}$$

These conditions are reduced to those for $N=2$ in the limit of a_3, $b_3 \to 0$.

Once the conditions (2.2b) have been established for large t, they hold for arbitrary t owing to (3.15). Furthermore, the conditions (2.2b) offer important information concerning the properties of zeros of (3.16). To see this, we refer to a famous theorem due to Hermite (Takagi 1965). Let

$$f(x) \equiv U(x) + \mathrm{i}V(x) \tag{3.30}$$

with

$$U(x) = x^N + \sum_{j=1}^{N} (-1)^j (a_j t + b_j) x^{N-j} \tag{3.31a}$$

$$V(x) = \sum_{j=1}^{N} (-1)^j a_j x^{N-j}. \tag{3.31b}$$

Hermite's theorem states that, if the signature of $\operatorname{Im} x_j$ is the same for all j, then the zeros of $U(x)=0$ and $V(x)=0$ are all real and simple and are isolated to each other. In order to clarify the latter meaning, let β_j be N distinct zeros of $U(x)=0$. It tells us that each zeros are ordered such as

$$\beta_1 < \alpha_1 < \ldots < \beta_{N-1} < \alpha_{N-1} < \beta_N. \tag{3.32}$$

In the present situation, β_j ($j=1, 2, \ldots, N$) are real functions of t while α_j ($j=1, 2, \ldots, N-1$) do not depend on t. On account of (3.31) and (3.32), the solution (2.35) can be rewritten in the form†

$$\theta = -2\tan^{-1} \frac{a_1 \Pi_{j=1}^{N-1} (x-\alpha_j)}{\Pi_{j=1}^{N} (x-\beta_j)}. \tag{3.33}$$

Asymptotic form of (3.33) for $t \to \pm\infty$ is found from (3.21)-(3.24) as

$$\theta \sim -2\tan^{-1}[a_1/(x - a_1 t - b_1 + a_2/a_1)] - 2\sum_{j=1}^{N-1} \tan^{-1}[\delta_j t^{-2}/(x - \alpha_j + \delta_j t^{-1})] \tag{3.34a}$$

with

$$\delta_j = \frac{\alpha_j^N + \Sigma_{k=1}^{N} (-1)^k b_k \alpha_j^{N-k}}{\Sigma_{k=1}^{N-1} (-1)^k (N-k) a_k \alpha_j^{N-k-1}} \qquad j = 1, 2, \ldots, N-1 \tag{3.34b}$$

† If we use the Hermite theorem, the solution (2.31) (or (3.33)) is derived quite simply as follows. Equations (3.30) and (2.6) imply $U_t/V + V_t/U = 1$ and since the zeros of U and V are different (Hermite theorem) the only possibility is that $U_t = (1-\lambda)V$ and $V_t = \lambda U$. Finally, from the fact that the coefficient of x^N in $f(x, t)$ is unity, there follows that $\lambda = 0$ and therefore that $f_t = \operatorname{Im} f$, namely equation (2.31). The author thanks one of the referees for this useful comment.

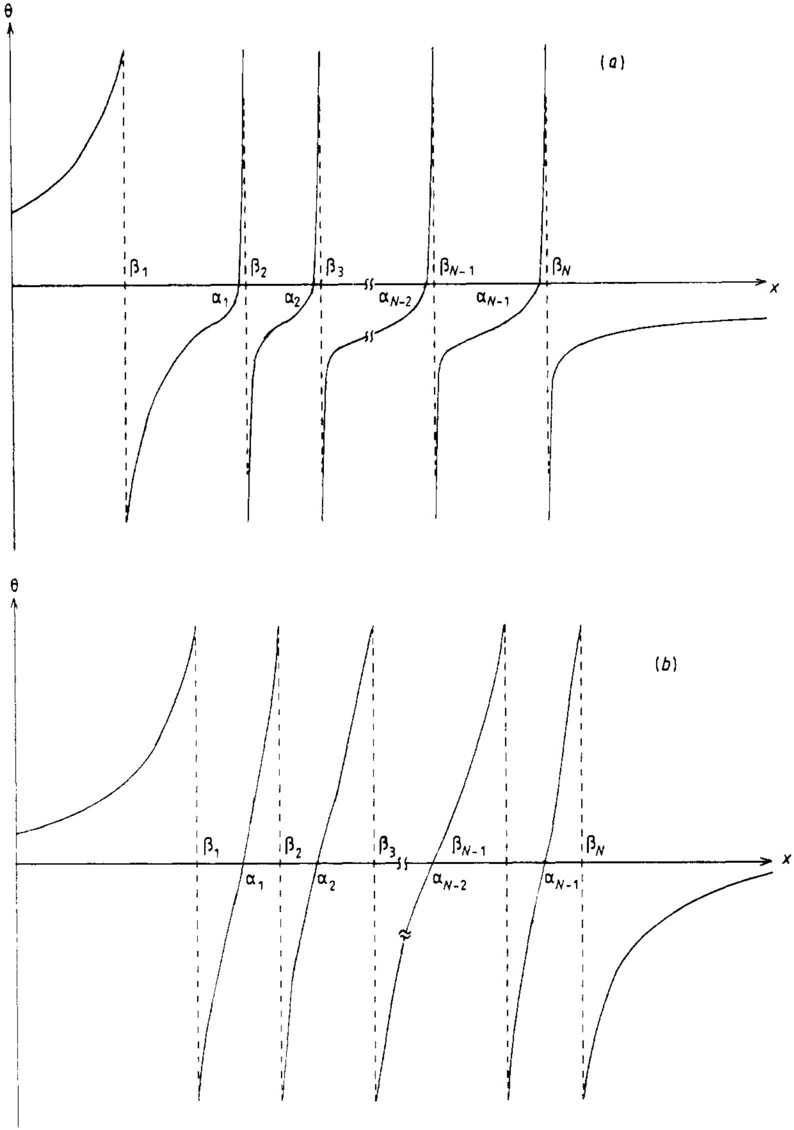

Figure 5. Rough profile of the solution (2.35) for three different time regions. In the figure, the positions of β_j ($j=1,2,\ldots,N$) are indicated by broken lines while those of the α_j ($j=1,2,\ldots,N-1$) are given by the points at which the full line representing the solution (2.35) intersects the x axis. (a) Large negative time region, (b) intermediate time region, (c) large positive time region.

and the x derivative of the expression (3.34a) gives

$$\theta_x \sim 2a_1/[(x-a_1t-b_1+a_2/a_1)^2+a_1^2] + \sum_{j=1}^{N-1} 2\delta_j t^{-2}/[(x-\alpha_j+\delta_j t^{-1})^2+(\delta_j t^{-2})^2]$$

$$\sim 2a_1/[(x-a_1t-b_1+a_2/a_1)^2+a_1^2]+2\pi\sum_{j=1}^{N-1}\delta(x-\alpha_j) \tag{3.35}$$

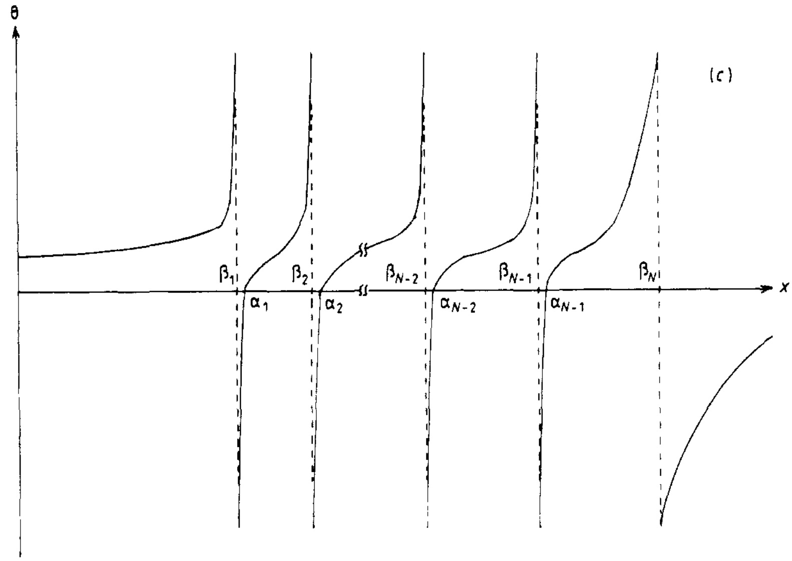

Figure 5. (continued)

with the aid of the formula

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{x^2+\varepsilon^2}=\pi\delta(x)\qquad (\delta(x)\text{: Dirac's delta function}). \tag{3.36}$$

The expression (3.35) expresses the pulse-like nature of the solution more clearly than (3.34*a*).

Consequently, an overall profile of the solution (2.35) may be pictured as follows (see also figures 5(*a*)-(*c*)).

For $t\to-\infty$, it is composed of a pulse moving in the positive x direction with a velocity a_1 and $N-1$ static pulses with very narrow widths located near $x=\alpha_1$, $x=\alpha_2,\ldots,x=\alpha_{N-1}$, respectively (see figure 5(*a*)). After a lapse of time, these $N-1$ static pulses grow wider and wider and eventually they behave like $N-1$ pulses moving in the positive x direction with velocities $\dot\beta_2$, $\dot\beta_3,\ldots,\dot\beta_N$ and centre positions being placed on $x=\beta_2$, $x=\beta_3,\ldots,x=\beta_N$, respectively, while a moving pulse behaves like a pulse with a velocity $\dot\beta_1$ and centre position being placed on $x=\beta_1$ (see figure 5(*b*)). Therefore, in this situation the solution is considered to be a superposition of N moving pulses. For $t\to+\infty$, a pulse with a velocity $\dot\beta_N$ behaves like a pulse with a constant velocity a_1. On the other hand, other $N-1$ pulses approach to $N-1$ immovable points $x=\alpha_1$, $x=\alpha_2,\ldots,x=\alpha_{N-1}$, respectively, and eventually they become to be static pulses with very narrow widths (see figure 5(*c*)). Thus, we have completed the detailed description on the behaviour of the solution (2.35). The solution has a quite different asymptotic form when it is compared with that of the soliton-type solution. This situation may be clarified by observing the asymptotic expression (3.35) of θ_x. It represents the superposition of a moving pulse with a Lorentzian profile and a train of $N-1$ static pulses with delta function profiles. On the other hand, the asymptotic form of the N-soliton solution of the BO equation (Matsuno 1984), for example, is expressed as

$$u \sim \sum_{j=1}^{N} 2a_j/[(x-a_j^{-1}t-x_{0j})^2+a_j^2] \tag{3.37}$$

which represents N moving pulses with Lorentzian profiles. Furthermore, the amplitude of each pulse in (3.34a) is all the same and takes the value of π (remember that we are considering the principal value of θ). The asymptotic form (3.34a) also has a novel characteristic in comparison with (3.1). The first term on the right-hand side of (3.34a) has the same asymptotic form as (3.1) but it takes into account the effect of interactions between pulses which is represented by a 'phase shift', $-a_2/a_1$. Finally, it should be noted that when both a_N and b_N vanish, (2.35) is reduced to a solution which includes $N-1$ pulses.

4. Concluding remarks

In this paper, we have developed a systematic method for constructing exact solutions of the SH equation and examined the dynamical properties of solutions in detail. The characteristics of the solution presented here are quite different from those of the usual N-soliton solution. If one requires the solution $\theta(x, t)$ be continuous, it has a multistep shape from $\theta(-\infty, t)=0$ to $\theta(+\infty, t)=2N\pi$. Therefore, the solution (2.35) (or (3.33)) may be named 'kinks' on the analogy of the well known kink solutions of the sine-Gordon equation.

In connection with the soliton theory, it is interesting to find conservation laws, Bäcklund transformations, etc, of the SH equation and these problems are now being pursued.

Acknowledgments

The author would like to express his sincere thanks to Professor M Nishioka for continual encouragement. He also thanks the referees for their many useful comments.

Appendix. Proof of (2.28)

Let J_k be

$$J_k=\sum_{j=0}^{k-1}(-1)^j s_{k-j}[(k-j)c_{j+1}-p_{j+1}] \qquad k=1,2,\ldots,m. \tag{A1}$$

We shall prove (2.28) by a mathematical induction. For $k=1$

$$J_1=-s_0(c_1-p_1)=0 \tag{A2}$$

by (2.10b) and (2.15a). Assume (2.28) for $k=1, 2, \ldots, n (n<m)$, i.e.

$$J_k=0 \qquad k=1,2,\ldots,n. \tag{A3}$$

Then

$$\begin{aligned} J_{n+1}&=\sum_{j=0}^{n}(-1)^j s_{n+1-j}[(n+1-j)c_{j+1}-p_{j+1}] \\ &=\sum_{j=0}^{n-1}(-1)^{j+1}s_{n-j}[(n-j)c_{j+2}-p_{j+2}]+s_{n+1}[(n+1)c_1-p_1]. \end{aligned} \tag{A4}$$

Substituting the relations

$$c_{j+2}=\sum_{r=0}^{j}(-1)^{j+r}s_{j+1-r}c_{r+1}+(-1)^{j+1}s_{j+2} \tag{A5}$$

$$p_{j+2}=\sum_{r=0}^{j}(-1)^{j+r}s_{j+1-r}p_{r+1}+(-1)^{j+1}(j+2)s_{j+2} \tag{A6}$$

which are derived from (2.19) and (2.10a), respectively, into (A4) yields

$$J_{n+1}=\sum_{j=0}^{n-1}\sum_{r=0}^{j}(-1)^{r+1}s_{n-j}s_{j+1-r}[(n-j)c_{r+1}-p_{r+1}]+\sum_{j=0}^{n-2}(n-2j-2)s_{n-j}s_{j+2}. \tag{A7}$$

The first term on the right-hand side of (A7) becomes

$$\begin{aligned}\sum_{j=0}^{n-1}\sum_{j=r}^{n-1}&(-1)^{r+1}s_{n-j}s_{j+1-r}[(n-j)c_{r+1}-p_{r+1}]\\ &=\sum_{r=0}^{n-1}\sum_{j=0}^{n-r-1}(-1)^{r+1}s_{n-j-r}s_{j+1}[(n-j-r)c_{r+1}-p_{r+1}]\\ &=\sum_{j=0}^{n-1}s_{j+1}\sum_{r=0}^{n-j-1}(-1)^{r+1}s_{n-j-r}[(n-j-r)c_{r+1}-p_{r+1}]\\ &=-\sum_{j=0}^{n-1}s_{j+1}J_{n-j}\\ &=0\end{aligned} \tag{A8}$$

because of (A3). On the other hand, the second term on the right-hand side of (A7) obviously vanishes. Thus we have

$$J_{n+1}=0 \tag{A9}$$

which implies that (A3) holds for $k=n+1$, completing the proof of (2.28).

References

Ablowitz M J and Segur H 1981 *Solitons and Inverse Scattering Transform* (Philadelphia: SIAM)
Benjamin T B 1967 *J. Fluid Mech.* **29** 559
Bullough R K and Caudrey P J (ed) 1980 *Solitons. Topics in Current Physics* vol 17 (Berlin: Springer)
Calogero F and Degasperis A 1982 *Spectral Transform and Solitons I* (Amsterdam: North-Holland)
Degasperis A and Santini P M 1983 *Phys. Lett.* **98A** 240
Degasperis A, Santini P M and Ablowitz M J 1985 *J. Math. Phys.* **26** 2469
Dodd R K, Eilbeck J C, Gibbon J D and Morris H C 1983 *Solitons and Nonlinear Wave Equations* (New York: Academic)
Fokas A S and Ablowitz M J 1983 *Stud. Appl. Math.* **68** 1
Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 *Phys. Rev. Lett.* **19** 1095
Hirota 1971 *Phys. Rev. Lett.* **27** 1192
Joseph R I 1977 *J. Phys. A: Math. Gen.* **10** L225
Joseph R I and Egri R 1978 *J. Phys. A: Math. Gen.* **11** L97
Kodama Y, Ablowitz M J and Satsuma J 1982 *J. Math. Phys.* **23** 564
Kubota T, Ko D R S and Dobbs D 1978 *J. Hydraut.* **12** 157
Matsuno Y 1979a *J. Phys. A: Math. Gen.* **12** 619
—— 1979b *Phys. Lett.* **74A** 223
—— 1980 *J. Phys. A: Math. Gen.* **13** 1519

Matsuno Y 1984 *Bilinear Transformation Method* (New York: Academic)
—— 1986 *Phys. Lett.* **119A** 229
—— 1987 *Phys. Lett.* **120A** 187
Ono H 1975 *J. Phys. Soc. Japan* **39** 1082
Santini P M, Ablowitz M J and Fokas A S 1985 *Preprint 47* INS
Satsuma J and Ishimori Y 1979 *J. Phys. Soc. Japan* **46** 681
Takagi T 1965 *Lecture on Algebra* (Tokyo: Kyoritsu Shuppan) (in Japanese)